An Exact Solution for a Static Spherical Shell of Matter in Vacuum.

C. LEIBOVITZ

Tel-Aviv University, Physics Department - Tel-Aviv

(ricevuto il 7 Marzo 1968)

Summary. — An exact solution for the gravitational field in a static spherical shell of matter is given. The exact junction conditions in the case of a thin shell are deduced.

It is known [1] that, in the case of a spherical static perfect fluid and for a line element of the kind

$$\mathrm{d}s^2 = e^{\omega}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) + e^{\sigma}\,\mathrm{d}t^2\,, \tag{1}$$

the pressure is a decreasing function of the radius co-ordinate r. It is therefore not possible to have a spherical shell of perfect fluid in equilibrium in vacuum since, in this case, the pressure is to be zero for two different values of r. However it is still possible to have a spherical shell of matter in equilibrium provided we allow for $T_1^1 \neq T_2^2$; in this case, the junction conditions [2] for the spherical shell with vacuum do not impose conditions on T_2^2; it is however necessary that we have $T_1^1 = 0$ for two different values $r = a$ and $r = b$ of the radius co-ordinate. This may be achieved if we have

$$T_1^1 = f(r)(r-a)(r-b)\,, \tag{2}$$

where $f(r)$ is a regular nonzero function in the interval $a < r < b$.

---

(1) B. HARRSION, K. THORNE, M. WAKANO and J. WHEELER: *Gravitational Theory and Gravitational Collapse*, Theorem 6 (Chicago, 1965), p. 26. (The theorem deals with cold catalysed matter but is in fact more general.)

(2) W. ISRAEL: *Proc. Roy. Soc.*, **248** A, 404 (1958).

Einstein equations for the static case may be written [3]

$$(3)\qquad -8\pi T_1^1 = e^{-\omega}\left(\frac{\sigma'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}\,,$$

$$(4)\qquad -8\pi T_2^2 = e^{-\omega}\left(\frac{\sigma''}{2} - \frac{\omega'\sigma'}{4} + \frac{\sigma'^2}{4} + \frac{\sigma' - \omega'}{2r}\right),$$

$$(5)\qquad 8\pi\varrho = e^{-\omega}\left(\frac{\omega'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi T_4^4\,.$$

That is to say three equations relating the five functions T_1^1, T_2^2, T_4^4, ω, σ. It is therefore possible to consider two of these quantities (or combinations of them) as arbitrary functions. However, in order to obtain physically meaningful solutions, the mathematically arbitary functions must be restricted somehow so that the calculated pressures and density satisfy some general physical conditions ($\varrho > T_1^1$, $\varrho > T_2^2$). In the case the arbitrary functions have Newtonian defined equivalents, it may be expected that if these functions are physically acceptable from the Newtonian point of view they will lead to general relativistic acceptable results (at least for these values of the paramters corresponding to weak fields).

We chose for arbitrary functions the radial pressure $p_r = -T_1^1$ and the Newtonian defined gravitational field intensity

$$(6)\qquad g = -\frac{\int_0^r 4\pi\varrho r^2}{r^2}\,.$$

From (3) and (5) we deduce

$$(7)\qquad \sigma' = \frac{8\pi p_r - 2g}{1 + 2gr}\,,$$

$$(8)\qquad e^{\omega} = (1 + 2gr)^{-1}\,.$$

It is to be remarked that eq. (7) and (8) derived here for the static case are still valid for the time-dependent case since eqs. (3) and (5) have the same form in the static and in the time-dependent case (for a line element of the form (1)).

Equations (7) and (8) are, formally, a solution of Einstein's equations including two arbitrary functions $p_r(r)$ and $g(r)$.

We will choose for the expression of g the Newtonian one corresponding to a homogeneous density

$$(9)\qquad \varrho = 3d/(4\pi) \qquad\qquad (d = \text{constant})$$

---

[3] R. Tolman: *Relativity Thermodynamics and Cosmology* (Oxford, 1934), p. 244.

and obtain for g the expression

$$g(r) = dr - da^3/r^2 = \frac{d}{r^2}(r^2 - a^3) \tag{10}$$

(a being the internal radius of the shell) and for e^{ω}

$$e^{\omega} = \left(1 + \frac{2da^3}{r} - 2dr^2\right)^{-1}. \tag{11}$$

We see from (10) that $g(a) = 0$, so that from (7) we obtain that $\sigma'(a) = 0$ if we want to have $p_r(a) = 0$.

We found that relatively simple results could be obtained by taking

$$\sigma' = \frac{k(r-a)}{r(u-r)} \qquad (k \text{ and } u \text{ being constants})$$

and restricting the values of k and u so that p_r is zero for $r = b$. The resulting solution is

$$\left\{\begin{aligned} e^{\omega} &= \left(1 - 2r^2d + \frac{2da^3}{r}\right)^{-1}, \\ e^{\sigma} &= qr^{ka/u}(u-r)^{k(1-a/u)}, \end{aligned}\right. \tag{12}$$

with

$$u = b + k(b-a) - \frac{kb(b-a)}{2d(b^3-a^3)}; \tag{13}$$

calculations give

$$\varrho = \frac{3d}{4\pi}, \tag{14}$$

$$8\pi p_r = \frac{(r-a)(b-r)}{r^3(u-r)}\left[\frac{k(a^2-br)}{a^2+b^2+ab} - 2d(k+1)(r^2+a^2+ar)\right], \tag{15}$$

$$\begin{aligned} -8\pi T^2 = \frac{1}{4}\left(1 - 2r^2d + \frac{2da^3}{r}\right)\Bigg[&\frac{2k(2ar-au-r^2)}{r^2(r-u)^2} + \frac{k^2(r-a)^2}{r^2(r-u)^2} - \\ &- \frac{2}{r}\,\frac{4rd + 2da^3/r^2}{1-2r^2d+2da^3/r} - \frac{k(r-a)}{r(r-u)}\cdot\frac{4rd+2da^3/r^2}{1-2r^2d+2da^3/r} + \frac{2k}{r^2}\,\frac{r-a}{r-u}\Bigg]. \end{aligned} \tag{16}$$

We have to impose the restriction $e^{\omega} > 0$ or

$$1 - \frac{2d}{r}(r^3 - a^3) > 0 \qquad \text{with } a < r < b, \tag{17}$$

this will be satisfied if

$$1-\frac{2d}{b}(b^3-a^3)>0\ .$$

However, from (13) we have

$$1-\frac{2d}{b}(b^3-a^3)=\frac{2d(b^3-a^3)(b-u)}{k(b-a)b}\ , \tag{18}$$

so that condition (17) is fulfilled if $(b-u)/k>0$.

It is convenient to take $k<0$ so that we must have also $u>b$ and therefore $(u-r)^{-1}$ will not contribute to infinities in the expression (15) for p_r in the range of r $a<r<b$.

Let us suppose that

$$\frac{2d(b^3-a^3)}{b}\ll 1 \tag{19}$$

and that

$$(b-a)/a\ll 1\ . \tag{20}$$

We may in this case write instead of (13)

$$u-b\approx -k/(6db) \tag{21}$$

and instead of (15)

$$8\pi p_r\approx\frac{(r-a)(b-r)}{r^3(u-r)}[-6db^2(k+1)]\approx 36d^2\frac{(k+1)}{k}(r-a)(b-r)\ . \tag{22}$$

Equation (22) shows that the radial pressure is at most of the order of

$$\frac{(b-a)^2}{b^2}d$$

and is positive for $k<-1$.

As for the nonradial pressure as given by (16), there are between the square brackets three terms containing explicitly the factor $(r-a)$ which are at most of the order $(b-a)d/a$; as to the two other terms

$$\frac{2k(2ar-au-r^2)}{r^2(r-u)^2}-\frac{2}{r}\,\frac{4rd+(2da^3/r^2)}{1-2r^2d+(2da^3/r)}\,,$$

we may write them within the approximation ($a \approx b \approx r$; $r - u \approx b - u \approx$ $\approx k/6bd$)

$$\frac{2k}{r(r-u)} - 12d \approx 0\,, \tag{23}$$

that is to say that T_2^2 is of order $(b-a)d/b$ at most. This means that our exact solution yields pressures as low as desired provided the inequalities (19) and (20) are satisfied. The solution may be joined smoothly to a Schwarzschild solution for the exterior and with a Minkowskian solution in the hole according to

$$e^{\omega} = \begin{cases} 1 & \text{for } r < a\,, \\ \left(1 - 2r^2 d + \dfrac{2da^3}{r}\right)^{-1} & \text{for } a < r < b\,, \\ \left(1 - \dfrac{2d(b^3 - a^3)}{r}\right)^{-1} & \text{for } r > b\,; \end{cases} \tag{24}$$

$$e^{\sigma} = \begin{cases} q a^{(k/au)} (u-a)^{k(1-a/u)} & \text{for } r < a\,, \\ q r^{(ka/u)} (u-r)^{k(1-a/u)} & \text{for } a < r < b\,, \quad (q = \text{constant}). \\ \dfrac{q b^{(ka/u)} (u-b)^{k(1-a/u)}}{1 - 2d(b^3 - a^3)/b} \left(1 - 2d\,\dfrac{b^3 - a^3}{r}\right) & \text{for } r > b\,, \end{cases} \tag{24'}$$

The ratio of the rate of a clock in the hole to the rate of a clock at infinity is given by

$$(b/a)^{-ka/2u} \left(\frac{u-b}{u-a}\right)^{-k(1-a/u)/2} \left[1 - \frac{2d(b^3 - a^3)}{b}\right]^{\frac{1}{2}}. \tag{25}$$

In the case $(b-a)/b \ll 1$ and $2d(b^3 - a^3)/b \ll 1$ the rate of the two clocks differs approximately by the Schwarzschild factor

$$\left[1 - \frac{2d(b^3 - a^3)}{b}\right]^{\frac{1}{2}}.$$

From the exact solution (24), we can investigate the limiting case of an infinitly thin shell of radius a. Writing

$$d(b^3 - a^3) = m$$

we have

$$e^{\omega} = \begin{cases} 1 & \text{for } r < a\,, \\ (1 - 2m/r)^{-1} & \text{for } r > a\,; \end{cases} \tag{26}$$

$$e^{\sigma} = \begin{cases} q a^{(ka/u)} (u - a)^{k(1-a/u)} & \text{for } r < a\,, \\ \dfrac{q a^{(ka/u)} (u - a)^{k(1-a/u)}}{1 - 2m/a} (1 - 2m/r) & \text{for } r > a\,. \end{cases} \tag{26'}$$

It is therefore seen that e^{σ} is continuous while e^{ω} is discontinuous across the shell.

It has been claimed in a recent study (4) that the metric of a thin shell would be continuous across the shell; our result shows that, in our system of co-ordinates at least, this is not true; more will be said on this point in a coming paper.

The solution (26) for an infinitely thin shell may be written in a more convenient way; if we put

$$q a^{(ka/u)} (u - a)^{k(1-a/u)} = 1 - 2m/a\,, \tag{27}$$

we obtain

$$e^{\omega} = \begin{cases} 1 & \text{for } r < a\,, \\ (1 - 2m/r)^{-1} & \text{for } r > a\,; \end{cases} \tag{28}$$

$$e^{\sigma} = \begin{cases} (1 - 2m/a) & \text{for } r < a\,, \\ (1 - 2m/r) & \text{for } r > a\,. \end{cases} \tag{28'}$$

(4) A. Papapetrou and A. Hamaoui: *Ann. Inst. Henri Poincaré*, **6**, No. **4** (1967).

RIASSUNTO (*)

Si dà la soluzione esatta del campo gravitazionale in uno strato sferico statico di materia. Si deducono le condizioni di giunzione esatte nel caso di uno strato sottile.

(*) *Traduzione a cura della Redazione.*

Точное решение для статической сферической оболочки вещества в вакууме.

Резюме (*). — Приводится точное решение для гравитационного поля в статической сферической оболочке вещества. Выводятся точные условия сшивания в случае тонкой оболочки.

(*) *Переведено редакцией.*